stochastic processes and their applications

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Optimal singular control strategies for controlling a process to a goal

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Abstract

An investor starting with initial wealth $z_0 > 0$ would like to achieve a total wealth a where $a > z_0$ before going bankrupt. The strategy is to allocate his wealth between a chosen risky asset and a bank account. The amount invested in the risky asset is given by an Itô process with infinitesimal parameters μ and σ. At time t, the choice of the risky asset is represented by $\mu(t)$ and $\sigma(t)$, which comes from a control set. This control set depends on the investor's wealth in the risky asset. At any time wealth can be transferred between the risky asset and the bank account without any transaction fee as long as the transaction process is of bounded variation. The problem considered here is to find an optimal strategy, which consists of an optimal choice of a risky asset and allocation of wealth, to maximize the probability of reaching a total wealth of a. © 1999 Elsevier Science B.V. All rights reserved.

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1. Introduction

Consider a stochastic process (X, Y) satisfying

$$\begin{aligned} X(t) &= x + \int_0^t \mu(s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}W(s) + A(t), \\ Y(t) &= y - A(t), \end{aligned} \tag{1.1}$$

where $\{W(t),\ t\geqslant 0\}$ is a standard Brownian motion on some probability space $(\Omega, \mathscr{F}, P)$ adapted to a right-continuous filtration $\{\mathscr{F}_t,\ t\geqslant 0\}$ and each $\mathscr{F}_t$ is contained in $\mathscr{F}$, is

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independent of $\{W(t+s)-W(t),\ s\geqslant 0\}$ and contains all P-null sets. The processes $\mu(t)$ and $\sigma(t)$ are real-valued progressively measurable and satisfy

$$\int_0^t (|\mu(s)|+\sigma^2(s))\,\mathrm{d}s<\infty \quad \text{a.s.} \tag{1.2}$$

for every $t>0$. The process $A(t)$ is assumed to be of bounded variation on finite intervals, right-continuous and adapted to $\{\mathscr{F}_t\}$.

In our optimal control problem, the processes $\mu(\cdot),\sigma(\cdot)$ and $A(\cdot)$ are considered as control variables or "strategies". The possible choices of μ and σ are determined by a collection $\{C(x')\colon 0\leqslant x'\leqslant a\}$ of non-empty subsets of $R\times R^+$. We assume that $(\mu(t),\sigma(t))$ belongs to $C(X(t-))$ for all $t>0$. (For convenience, we have assumed that A and consequently the processes X and Y are right-continuous.) We restrict, our attention to those strategies $\mu(\cdot),\sigma(\cdot)$ and $A(\cdot)$ which satisfy the above assumptions and that yield the controlled processes $X(t)$ and $Y(t)$ taking values in an interval $[0,a]$ for a fixed $a>0$. Furthermore, it is assumed that $x+y<a$.

Let $\Sigma(x,y)$ be the collection of all the controlled processes (X,Y) described above which are available to a controller with initial data $(X(0),Y(0))=(x,y)$ where $x\geqslant 0$, $y\geqslant 0$ and $x+y<a$. Our objective here is to maximize the probability that $X+Y$ reaches a before 0. Hence for an available (X,Y) in $\Sigma(x,y)$, we introduce the process Z by

$$Z(t)=X(t)+Y(t) \quad \text{for all } t\geqslant 0. \tag{1.3}$$

The value function for the control problem is given by

$$V(x,y)=\sup\{P[Z \text{ reaches } a \text{ before } 0]\colon (X,Y)\in\Sigma(x,y)\}, \tag{1.4}$$

where $[Z$ reaches a before $0]$ is the event $[Z(t)\geqslant a$ and $\inf_{[0,t]} Z(s)>0$ for some $t\geqslant 0]$.

To motivate this problem, consider an investor, with initial wealth z, who intends to achieve a financial goal of a ($a>z$) before bankruptcy. At any time instant $t\geqslant 0$, the investor has two investment options: The first option is a bank account which pays zero interest. The second option is to select exactly one risky asset from a collection of available risky assets. For example, the set of available risky assets could be the collection of all portfolios available by investing in a finite set of stocks. However, this set of available risky assets depends on the amount invested in the risky assets at the time. Let $X(t)$ and $Y(t)$ be the amount in the risky investment and in the bank account at time t, respectively. Notice that $X(0)+Y(0)=z$ and the collection of available risky assets at time t is represented by the control set $C(X(t-))$. Furthermore, the investor can transfer the funds between the bank account and the risky investment without any transaction cost. The bounded variation process $A(t)$ represents the cumulative amount transacted from the bank account to the risky investment during $[0,t]$. We consider only those investment processes (X,Y) in $\Sigma(x,y)$ with initial investment $(X(0),Y(0))=(x,y)$ such that $x+y=z$. The process Z given in (1.3) represents the total wealth of the investor. Hence, the optimal investment decision problem is to choose a risky asset and to find optimal buying and selling policies of this asset to maximize the probability that the total wealth process Z reaches a before going bankrupt.

The key to the definition of the optimal process is the function

$$\rho(x) = \sup\{\mu/\sigma^2 \colon (\mu, \sigma) \in C(x)\} \quad \text{for } 0 \leqslant x \leqslant a. \tag{1.5}$$

(Here 0/0 is taken to be $-\infty$.) In a number of related optimal control problems (Pestien and Sudderth, 1985, 1988; Sudderth and Weerasinghe, 1991,1992; Weerasinghe, 1998), it has been proved optimal to choose the controls (μ, σ) at each x so that μ/σ^2 attains the supremum $\rho(x)$. It remains true in the problem we consider here. Throughout this article, we assume that ρ can be written in the form

$$\rho(x) = \mu_0(x)/\sigma_0(x)^2 \quad \text{for } 0 \leqslant x \leqslant a, \tag{1.6}$$

where μ_0 and σ_0 are continuous functions on $[0, a]$, $\inf_{[0,a]} \sigma_0(x) > 0$ and $(\mu_0(x), \sigma_0(x)) \in C(x)$ for each x in $[0, a]$. The functions μ_0 and σ_0 will be used to select the optimal μ and σ. The optimal choice for the process A is intimately related to the behavior of ρ on $[0, a]$. For certain ρ, there is no optimal process.

The problem considered in this article is closely related to the work in Sudderth and Weerasinghe (1991, 1992) and Weerasinghe (1998). In Sudderth and Weerasinghe (1991, 1992), the same optimization problem is considered, but the investor is only allowed to withdraw the wealth from the bank account as opposed to our problem, where we allow transactions between the two assets. Explicitly, in Sudderth and Weerasinghe (1991, 1992) it is assumed that $A(\cdot)$ is a non-decreasing process while here we only assume that $A(\cdot)$ is of bounded variation. In the case of ρ monotone decreasing, a related optimization problem with proportional transaction costs is solved in Weerasinghe (1998). In our paper, optimal strategies do not exist (only ε-optimal strategies exist) for this case while in Weerasinghe (1998) new optimal strategies are derived in the presence of transaction costs.

In Pestien and Sudderth (1985,1988) considered an important class of optimal control problems related to controlling a process to a goal and their relation to continuous-time casinos. In those-articles, there is no risk-free bank account available to the investor and hence all the wealth is invested in a risky investment. Hence, in Pestien and Sudderth (1985, 1988), the investor does not have available the control process $A(t)$, which is in our problem.

Due to the availability of the control process $A(\cdot)$, the investor in our problem has a much larger collection of available strategies. It also happens here that when ρ is monotone increasing, the optimal strategy for our problem coincides with that of Pestien and Sudderth (1985, 1988) and Sudderth and Weerasinghe (1991), but this is the only case in which this happens. In this case it is optimal to set $A(0+) = y$ and $A(t) = y$ for $t > 0$ and hence making $Y(t) \equiv 0$ for all $t > 0$. We call this "Bold play" strategy and discuss it in Corollary 4.3.

For a given (x, y) in $\Sigma(x, y)$ the total wealth process Z defined in (1.3) has continuous paths and satisfies

$$Z(t) = (x + y) + \int_0^t \mu(s)\,\mathrm{d}s + \int_0^t \sigma(s)\,\mathrm{d}W(s), \tag{1.7}$$

where $(\mu(t), \sigma(t)) \in C(X(t-))$, while in Pestien and Sudderth (1985, 1988), it is assumed that $(\mu(t), \sigma(t)) \in C(Z(t))$ for each $t > 0$.

We always select μ_0 and σ_0 given in (1.6) as our candidates for optimal drift and diffusion terms. Our basic principle for choosing an optimal (or ε-optimal) buying and selling policy $A(\cdot)$ for the risky investment is the following: Introduce the function ρ^* on $[0, a]$ by $\rho^*(x) = \sup_{[0,x]} \rho(u)$. At any time instant t, if the total wealth is equal to $Z(t)$, then we would like to control the $A(t)$ process so that the distance $|X(t) - \theta(Z(t))|$ is minimal, where the function θ is defined by $\theta(z) = \sup\{u\colon 0 \leqslant u \leqslant z,\ \rho(u) = \rho^*(z)\}$. If $X(t) \equiv \theta(Z(t))$ for all t, then it will give an optimal strategy. In general, it is not possible to find a buying and selling policy $A(\cdot)$ which makes $X(t)$ identically equal to $\theta(Z(t))$ and in this case there will not be an optimal strategy. For example, if $\rho(0) > \rho(x)$ for all $x > 0$, then clearly $\theta(z) = 0$ for all z, and hence $\theta(Z(t)) \equiv 0$ for all t. But it is not possible to find a bounded variation process $A(\cdot)$ so that the X process satisfies (1.1) and $X(t) \equiv \theta(Z(t))$ for all t.

In Section 2, we provide an application to illustrate the results in this article. Similar to many articles in the optimal control literature (for a Reference Fleming and Rishel, 1975; Fleming and Soner, 1993) we formulate a verification lemma in Section 3. We use it to obtain a closed form expression for the value function.

There are three main theorems proved in Section 4. In Theorem 4.2, we show that a given admissible process (X, Y) in $\Sigma(x, y)$ is an optimal process if and only if the equalities $\rho^*(X(t) + Y(t)) = \rho(X(t-)) = \rho(t)$ hold with respect to the product measure $\lambda \otimes P$ on $[0, \infty) \times \Omega$ where λ is the Lebesque measure on $[0, \infty)$, P is the probability measure on the probability space Ω and the function ρ^* is given by $\rho^*(z) = \sup\{\rho(u)\colon 0 \leqslant u \leqslant z\}$ for each z in $[0, a]$. Theorem 4.5 gives a sufficient condition in terms of the function ρ, which guarantees the non-existence of an optimal process. Finally, in Theorem 4.7, we derive a sufficient condition in terms of the function ρ for the existence of an optimal process. In the proof of this theorem we construct an optimal process. In Section 5, we consider classes of functions ρ in which no optimal process exists. For these cases, we illustrate how to use the reflecting diffusions and local time processes to construct ε-optimal processes.

2. An application

Example 1. Consider an investment model with one risky asset, which we call a stock, and a risk free bank account. The interest rate for the bank account is zero. Let $X_1(t)$ be an investor's holding in the stock at time $t \geqslant 0$ and $Y_1(t)$ be the amount in the investor's bank account at time $t \geqslant 0$. The investor begins with an initial endowment $(X_1(0), Y_1(0)) = (x, y)$. The funds can be transferred between two assets without a transaction cost and the cumulative transactions during $[0, t]$ is considered as a process of bounded variation.

It is assumed that there is a constant $\delta > 0$ which represents the minimum amount to be kept in the risky asset. Furthermore no loans are allowed from the bank account. Hence we assume that $X_1(t) \geqslant \delta$ and $Y_1(t) \geqslant 0$ and the state $(\delta, 0)$ is considered as the bankruptcy state for the joint process (X_1, Y_1). If $X_1(t) = \delta$ and $Y_1(t) > 0$ for some t, then it is necessary to transfer funds from Y_1 to keep X_1 above the value δ. Similar to the problems studied in Fleming and Soner (1993) and Shreve and Soner (1994), we use the standard financial model, i.e. $\mu(t) = \mu_0 X_1(t)$ and $\sigma(t) = \sigma_0 X_1(t)$ in Eq. (1.1), to represent $X_1(t)$ and $Y_1(t)$. More precisely,

$$\begin{aligned} X_1(t) &= x + \int_0^t \mu_0 X_1(s)\,\mathrm{d}s + \int_0^t \sigma_0.X_1(s)\,\mathrm{d}W(s) + A(t), \\ Y_1(t) &= y - A(t), \end{aligned} \tag{2.1}$$

where the process $A(\cdot)$ is right-continuous and is of bounded variation on finite intervals. $A(t)$ represents the cumulative transactions from the bank account to the risky asset. The constants μ_0 and σ_0 are known to the investor.

Let g be the financial goal of the investor. The investor's total wealth at time t is given by the process $Z_1(t) = X_1(t) + Y_1(t)$. (It is assumed that $g > x + y$.) The only available control is the buying and selling policy process $A(\cdot)$. The investor's objective is to maximize the probability that the total wealth $Z(t)$ reaches the goal of g before bankruptcy.

To make use of our results in this paper for this problem, we introduce the processes X and Y by $X(t) = X_1(t) - \delta$ and $Y(t) \equiv Y_1(t)$. Hence,

$$\begin{aligned} X(t) &= x - \delta + \int_0^t \mu_0(X(s)+\delta)\,\mathrm{d}s + \int_0^t \sigma_0(X(s)+\delta)\,\mathrm{d}W(s) + A(t), \\ Y(t) &= y = A(t). \end{aligned} \tag{2.2}$$

Introduce $Z(t) = X(t) + Y(t) = Z_1(t) - \delta$ and $a = g - \delta$. Now (X, Y) agree with (1.1) and the problem described in the introduction. In this case, the collection $\{C(x')\colon 0 \leqslant x' \leqslant a\}$ of control sets is given by $C(x') = \{(\mu_0 x', \sigma_0 x')\}$ for each $0 \leqslant x' \leqslant a$. Hence the control problem is to find an optimal bounded variation process $A(\cdot)$ to maximize the probability $P[Z(t)$ reaches a before $0 | (X(0), Y(0)) = (x - \delta, y)]$. To describe the optimal choices which follow from our results, we introduce $\rho_0 = \mu_0/\sigma_0^2$ and the function $\rho(x)$ defined in (1.5) is given by

$$\rho(x) = \frac{\rho_0}{(x+\delta)} \quad \text{for } 0 \leqslant x \leqslant a. \tag{2.3}$$

If $\mu_0 > 0$ then $\rho(x)$ is strictly decreasing on $[0, a]$. Therefore, we can apply Theorem 5.4 to conclude that there is no optimal strategy. But we derive a sequence of ε-optimal strategies for the buying and selling policy in Example 1 of Section 5.

In the case $\mu_0 \leqslant 0$, then $\rho(x)$ is monotone increasing on $[0, a]$. Therefore, we can apply Theorem 4.2, and its Corollary 4.3 to conclude that the "bold play" strategy of taking $A(0+) = y$, $A(t) \equiv y$ for all t, is an optimal buying and selling policy.

3. The value function

For a given $(X, Y) \in \Sigma(x, y)$, we consider the set $E = \{(x, y)\colon x \geqslant 0, y \geqslant 0, x + y \leqslant a\}$ to be the state space. The following verification lemma helps us to compute the value function and also to determine the optimal strategies.

The proof of this lemma utilizes Itô's formula for general semi-martingales (Meyer, 1974, p. 285, also Shreve and Soner, 1994, p. 625, formula (4.7)), and it is similar to that of Lemma 2.1 in Sudderth and Weerasinghe (1991). Therefore, we omit the proof.

Lemma 3.1. *Let Q be a real-valued function defined on an open subset G of R^2 containing E. Assume*

(i) *Q has continuous second-order derivatives on G.*

(ii) *$0 \leqslant Q \leqslant 1$ on $E, Q(0,0)=0$ and $Q(x,y)=1$ on $x+y=a$.*

(iii) *The following inequalities hold on the set $E \cap \{(x,y)\colon 0 < x+y < a\}$*

(a)

$$\frac{\partial^2 Q}{\partial x^2}(x,y) + 2\rho(x)\frac{\partial Q}{\partial x}(x,y) \leqslant 0$$

and

(b)

$$\frac{\partial Q}{\partial x}(x,y) = \frac{\partial Q}{\partial y}(x,y) \geqslant 0.$$

Then $Q(x,y) \geqslant V(x,y)$ for all $(x,y) \in E$.

Remark. Condition (iii)(b) together with (i) and (ii) implies that there is a real-valued twice differentiable function g: $[0,a] \to [0,1]$ such that g is monotone increasing, $g(0)=0$, $g(a)=1$ and $Q(x,y)=g(x+y)$ for each (x,y) in E.

Our next proposition gives a candidate for Q in the verification lemma. It turns out this candidate is equal to the value function.

For the function ρ given in (1.5), define

$$\rho^*(x) = \sup\{\rho(u)\colon 0 \leqslant u \leqslant x\}. \tag{3.1}$$

Since ρ is assumed continuous in (1.6), ρ^* will be continuous and monotone increasing. Now let $S^*(\cdot)$ be the scale function of ρ^* which is given by

$$S^*(x) = \int_0^x \mathrm{e}^{-2\int_0^u \rho^*(r)\,\mathrm{d}r}\,\mathrm{d}u. \tag{3.2}$$

Proposition 3.2. *Let $Q^*(x,y) = S^*(x+y)/S^*(a)$, then $Q^*(x,y) \geqslant V(x,y)$ for all $(x,y) \in E$.*

Proof. We take $Q(x,y) \equiv Q^*(x,y)$ in Lemma 3.1. Clearly Q^* is C^2 in (x,y). The conditions (i), (ii) and (iii)(b) are easy to check and it remains to verify (iii)(a).

Since S^* is increasing in x and $S^*(a) \geqslant 0$, it follows that $(\partial Q^*/\partial x)(x,y) \geqslant 0$.

For $(x,y) \in E$, $\quad \rho(x) \leqslant \rho^*(x+y)$, hence for $(x,y) \in E$,

$$\frac{\partial^2 Q^*}{\partial x^2}(x,y) + 2\rho(x)\frac{\partial Q^*}{\partial x}(x,y) \leqslant \frac{\partial^2 Q^*}{\partial x^2} + 2\rho^*(x+y)\frac{\partial Q^*}{\partial x}.$$

But $S^*(x)$ satisfies $\partial^2 S^*/\partial x^2 + 2\rho^*(x)\partial S^*/\partial x = 0$, and therefore $\partial^2 S^*/\partial x^2(x+y) + 2\rho^*(x+y)\partial S^*/\partial x(x+y) = 0$ and consequently $\partial^2 Q^*/\partial x^2 + 2\rho^*(x+y)\partial Q^*/\partial x = 0$. This verifies iii(a) and hence the proof is complete. $\square$

To prove that the value function is indeed equal to $S^*(x+y)/S^*(a)$, we construct a sequence of ε-optimal strategies for a general continuous function ρ. As we explained in the introduction, it is reasonable to choose optimal drift and diffusion coefficients to be $\mu_0(\cdot)$ and $\sigma_0(\cdot)$. Therefore, what is left is to find a candidate for the optimal choice of the bounded variation process $A(\cdot)$. The following proposition can be used in construction of such a strategy.

Proposition 3.3. *Let* $\varepsilon \geqslant 0$ $(x,y) \in E$ *and* $(X_\varepsilon, Y_\varepsilon) \in \Sigma(x,y)$ *satisfy*

$$\begin{aligned} dX_\varepsilon(t) &= \mu_0(X_\varepsilon(t-)\,dt + \sigma_0(X_\varepsilon(t-))\,dW(t) + dA_\varepsilon(t),\\ dY_\varepsilon(t) &= -dA_\varepsilon(t), \end{aligned} \tag{3.3}$$

where $A_\varepsilon(\cdot)$ *is an adapted right continuous bounded variation process,* $\mu_0(\cdot)$ *and* $\sigma_0(\cdot)$ *are continuous functions satisfying* (1.6) *and* $\{W(t)\colon t \geqslant 0\}$ *is a Brownian motion.*

Define the total wealth process $Z^\varepsilon(\cdot)$ *by* $Z^\varepsilon(t) = X^\varepsilon(t) + Y^\varepsilon(t)$ *and* $Q_\varepsilon(x,y) = P[Z^\varepsilon(t)$ *reaches* a *before* 0 *for some* $t > 0$ $|(X^\varepsilon(0), Y^\varepsilon(0)) = (x,y)]$.

Assume

$$|\rho(X^\varepsilon(t)) - \rho^*(Z^\varepsilon(t))| \le \varepsilon \quad \textit{for every } t \geqslant 0, \tag{3.4}$$

where ρ *and* ρ^* *are given by* (1.6) *and* (3.1), *respectively.*

Then

(i) $|Q_\varepsilon(x,y) - S^*(x+y)/S^*(a)| \leqslant C\varepsilon$ *where* $S^*(\cdot)$ *is given by* (3.2) *and the constant* C *depends only on* μ_0, σ_0 and a.

(ii) *The process* $(X_\varepsilon, Y_\varepsilon)$ *is an* ε*-optimal process and hence the value function* $V(x,y)$ *is given by* $V(x,y) = S^*(x+y)/S^*(a)$.

Proof. Let $(X_\varepsilon, Y_\varepsilon) \in \Sigma(x,y)$ satisfying (3.3) and (3.4). Introduce

$$H(x,y) = \frac{S^*(x+y)}{S^*(a)} \quad \text{where } S^* \text{ is given by (3.2)} \tag{3.5}$$

and

$$\lambda = \inf\{t \geqslant 0\colon X_\varepsilon(t) + Y_\varepsilon(t) = 0 \text{ or } a\}. \tag{3.6}$$

Notice $Z_\varepsilon(0) = x + y$ and

$$dZ_\varepsilon(t) = \mu_0(X_\varepsilon(t-))\,dt + \sigma_0(X_\varepsilon(t-))\,dW(t).$$

Since $\sigma_0^2(\cdot)$ is a bounded continuous function satisfying $\sigma_0^2(x) > 0$ on $[0,a]$, standard arguments using Itô's formula yields $E(\lambda) < \infty$. Therefore, we can find a positive constant M satisfying

$$E\left(\int_0^\lambda \sigma_0^2(X_\varepsilon(t-))\,dt\right) \leqslant M < \infty. \tag{3.7}$$

To prove the theorem, we apply Itô's formula to $H(X_\varepsilon(t), Y_\varepsilon(t))$ given in (3.5). This calculation is similar to the proof of Lemma 2.1 in Sudderth and Weerasinghe (1991).

So we have

$$
\begin{aligned}
&E[H(X_\varepsilon(t\wedge\lambda), Y_\varepsilon(t\wedge\lambda))]\\
&\quad = H(x,y)\\
&\qquad + E\left[\int_0^{t\wedge\lambda} \frac{\sigma_0^2(X_\varepsilon(s-))}{2}\left[\frac{\partial^2 H}{\partial x^2} + 2\rho(X_\varepsilon(s-))\frac{\partial H}{\partial x}\right](X_\varepsilon(s-), Y_\varepsilon(s-))\,\mathrm{d}s\right]\\
&\quad = \frac{S^*(x+y)}{S^*(a)}\\
&\qquad + \frac{1}{2S^*(a)}E\left[\int_0^{t\wedge\lambda} \sigma_0^2(X_\varepsilon(s-))\left(\frac{\mathrm{d}^2}{\mathrm{d}x^2} + 2\rho(X_\varepsilon(s-))\frac{\mathrm{d}}{\mathrm{d}x}\right)S^*(Z_\varepsilon(s-))\,\mathrm{d}s\right].
\end{aligned}
$$

Hence,

$$
\begin{aligned}
&\frac{E[S^*(Z_\varepsilon(t\wedge\lambda))]}{S^*(a)}\\
&\quad = \frac{S^*(x+y)}{S^*(a)} + \frac{1}{S^*(a)}E\left[\int_0^{t\wedge\lambda} \sigma_0^2(X_\varepsilon(s-))[\rho(X_\varepsilon(s-)) - \rho^*(Z_\varepsilon(s))]S^{*\prime}(Z_\varepsilon(s))\,\mathrm{d}s\right].
\end{aligned}
$$

Letting $t\uparrow+\infty$ in the left-hand side we get

$$
Q_\varepsilon(x,y) = E\left(\frac{S^*(Z_\varepsilon(\lambda))}{S^*(a)}\right).
$$

Let $\gamma = \sup_{[0,a]}|\rho(x)|$. Then, by (1.6), γ is finite and $0 < S^{*'}(x) \leqslant \mathrm{e}^{2\gamma a}$ for all x in $[0,a]$. Hence, using (3.4) and (3.7) we obtain

$$
\left|Q_\varepsilon(x,y) - \frac{S^*(x+y)}{S^*(a)}\right| \leqslant \frac{M}{S^*(a)}\cdot \mathrm{e}^{2\gamma a}\cdot\varepsilon \equiv C\varepsilon,
$$

where $C = M\mathrm{e}^{2\gamma a}/S^*(a)$ is a positive constant.

This concludes the proof of (i).

For (ii), notice $V(x,y) \leqslant S^*(x+y)/S^*(a)$ from Proposition (3.2). Since $(X_\varepsilon, Y_\varepsilon) \in \Sigma(x,y)$, it also follows that

$$
Q_\varepsilon(x,y) \leqslant V(x,y).
$$

Now (ii) follows by letting ε tend to zero. □

Remark. It is easy to check that if $\varepsilon = 0$ in Proposition 3.3, then $(X_\varepsilon, Y_\varepsilon)$ gives an optimal process.

In our next theorem, we use Proposition 3.3 to show that the value function is equal to $S^*(x+y)/S^*(a)$ for the case of a general continuous ρ. In the proof given below, we also construct a sequence of ε-optimal processes.

Theorem 3.4. *Let ρ be continuous on $[0,a]$. Then the value function $V(x,y)$ defined in (1.4) is equal to $S^*(x+y)/S^*(a)$ where S^* is given by (3.2).*

Proof. For a given ρ continuous in $[0, a]$ we define the function ρ^* on $[0, a]$ as in (3.1).

Fix $\varepsilon > 0$. Then for some n it is possible to pick $(n+1)$ points $z_0, z_1, \dots, z_n$ from $[0, a]$, satisfying

(i) $0 = z_0 < z_1 < \cdots < z_n < a$,

(ii) $0 < \rho(z_{i+1}) - \rho(z_i) < \varepsilon$, $0 \leqslant i < n$, and

(iii) $\rho^*(z_i) = \rho(z_i)$ for each $i = 0, 1, 2, \dots, n$.

Now pick $\delta > 0$ such that

(i) $\delta < \min_{0 \leqslant i \leqslant n} |z_{i+1} - z_i|$, and

(ii) if $x, y \in [0, a]$ and $|x - y| < \delta$ then $|\rho(x) - \rho(y)| < \varepsilon$.

We introduce a sequence of disjoint closed intervals $I_0, I_1, \dots, I_n$ defined by $I_k = [z_k, z_k + \delta]$ for each $k = 0, 1, 2, \dots, n$. For a given initial wealth $z \in (0, a)$, we define the process $(X_\varepsilon, Y_\varepsilon) \in \Sigma(x, y)$ and satisfying (3.3) as follows:

If $z \in (z_k, z_k + \delta]$ for any $k = 0, 1, 2, \dots, n$ then we make $X_\varepsilon(0) = z$ and $Y_\varepsilon(0) = 0$.

Now $(X_\varepsilon(t), Y_\varepsilon(t))$ satisfies

$$\mathrm{d}X_\varepsilon(t) = \mu_0(X_\varepsilon(t)\,\mathrm{d}t + \sigma_0(X_\varepsilon(t))\,\mathrm{d}W(t) + \mathrm{d}L^k(t) - \mathrm{d}L^{k+}(t)$$

and

$$\mathrm{d}Y_\varepsilon(t) = \mathrm{d}L^{k+}(t) - \mathrm{d}L^k(t)$$

for $0 < t < \tau$ where $L^k(\cdot)$ is the local time of $X_\varepsilon(\cdot)$ at z_k, $L^{k+}(\cdot)$ is the local time of $X_\varepsilon(\cdot)$ at $z_k + \delta$ and τ is the first time $X_\varepsilon(t) + Y_\varepsilon(t) = z_k$ or $(z_{k+1} + \delta/2)$. Notice that up to the stopping time τ, $X_\varepsilon(\cdot)$ is a diffusion on the state space $[z_k, z_k + \delta]$ and is instantaneously reflecting at the points z_k and $z_k + \delta$. At the random time τ, if $X_\varepsilon(\tau-) + Y_\varepsilon(\tau-) = z_k$ then the process $(X_\varepsilon, Y_\varepsilon)$ jumps to $(X_\varepsilon(\tau), Y_\varepsilon(\tau)) = (z_{k-1} + \delta, z_k - (z_{k-1} + \delta))$ and similarly if $X_\varepsilon(\tau-) + Y_\varepsilon(\tau-) = z_{k+1} + \delta/2$, then it jumps to $(X_\varepsilon(\tau), Y_\varepsilon(\tau)) = (z_{k+1} + \delta/2, 0)$. Hence $X_\varepsilon(\tau)$ is inside one of the intervals $(z_k, z_k + \delta]$ for some k, and then the same procedure may be continued.

If the initial wealth z does not belong to the set $\bigcup_{j=0}^{n} (z_j, z_j + \delta)$, then there will be a maximum k such that $z_k + \delta < z$. Then we take $X_\varepsilon(0) = z_k + \delta$, $Y_\varepsilon(0) = z - (z_k + \delta)$ and follow the method described above. Now using the triangular inequality twice one can easily show that $|\rho(X_\varepsilon(t)) - \rho^*(Z_\varepsilon(t))| \leqslant 3\varepsilon$ and hence we can apply Proposition 3.3 and it completes the proof of the theorem. $\square$

Remark. The method described in the above theorem will not identify any optimal process even if there is one. In the next section, we will identify optimal processes for certain classes of ρ.

4. When does an optimal process exist?

In the last section, we identified a closed-form expression for the value function. Now we are concerned about the existence of an optimal process. Our discussion in this section gives positive and negative results along this direction. The following lemma will be used to obtain sufficient conditions on the behaviour of ρ for non-existence of an optimal process.

Lemma 4.1. *Let* $\{(X(t), Y(t)): t \geqslant 0\}$ *be an optimal process satisfying the equations* (1.1) *and let* τ *be the stopping time given by* $\tau = \inf\{t \geqslant 0: Z(t) = 0$ *or* $a\}$ *where* $Z(t) = X(t) + Y(t)$. *Then*

$$E\left[\int_0^\tau (\sigma^2(s)\rho^*(Z(s)) - \mu(s))\,\mathrm{d}s\right] = 0, \tag{4.1}$$

where ρ^* *is given by* (3.1) *and*

$$\rho^*(Z(t)) = \rho(X(t-)) = \rho(t), \ (\lambda \otimes P)\text{–a.s.}, \tag{4.2}$$

where λ *is the Lebesgue measure on* $[0, \infty)$.

Proof. From the results in the previous sections we know that the value function $V(x, y) = \frac{S^*(x+y)}{S^*(a)}$ where S^* is as in (3.2). Then using Itô's lemma we derive.

$$E\left[\frac{S^*(Z(t\wedge\tau))}{S^*(a)}\right] = \frac{S^*(x+y)}{S^*(a)}$$
$$+\frac{1}{S^*(a)}E\left[\int_0^{t\wedge\tau}\left(\frac{1}{2}\sigma^2(s)S^{*\prime}(Z(s)) + \mu(s)S^{*\prime}(Z(s))\right)\,\mathrm{d}s\right].$$

But S^* satisfies the differential equation $S^{*\prime\prime}(z) + 2\rho^*(z)S^{*\prime}(z) = 0$ for $0 < z < a$ and $P[\tau < t, Z(\tau) = a] \leqslant E[S^*(Z(t\wedge\tau))/S^*(a)]$. Since $(X(t), Y(t))$ is an optimal process, notice that $P[\tau < t, Z(\tau) = a]$ increases to $S^*(x+y)/S^*(a)$ as t tends to infinity. Therefore, we have

$$0 \leqslant \frac{1}{S^*(a)}E\left[\int_0^{t\wedge\tau}(\sigma(s)^2\rho^*(Z(s)) - \mu(s))S^{*\prime}(Z(s))\,\mathrm{d}s\right]$$
$$\leqslant \frac{S^*(x+y)}{S^*(a)} - P[\tau < t,\ Z(\tau) = a].$$

The first inequality is true since $\mu(s)/\sigma(s)^2 \leqslant \rho(X(s-)) \leqslant \rho^*(Z(s))$ and $S^{*\prime}$ is non-negative. The second inequality follows from the discussion above. Notice that the above integrand is non-negative, and $S^{*\prime}$ is bounded below on $[0, a]$ by a positive constant, and therefore letting t go to infinity, we have

$$E\left[\int_0^\tau (\sigma(s)^2\rho^*(Z(s)) - \mu(s))\,\mathrm{d}s\right] = 0.$$

Hence (4.1) follows.

Eq. (4.2) is an easy consequence of (4.1). From the assumptions on the control sets we have $\mu(s)/\sigma(s)^2 \leqslant \rho(X(s-)) \leqslant \rho^*(Z(s))$. Hence $\sigma(s)^2\rho^*(Z(s)) - \mu(s) \geqslant 0$ and using (4.1), it follows that $\sigma(s)^2\rho^*(Z(s)) = \mu(s)$ a.s. in $(\lambda \otimes P)$.

But $\rho^*(Z(s)) \geqslant \rho(X(s-)) \geqslant \rho(s) \equiv \mu(s)/\sigma^2(s)$ and therefore (4.2) follows. □

Our next theorem combines the results of Proposition 3.3 together with Lemma 4.1.

Theorem 4.2. *Let* $\{(X(t), Y(t)): t \geqslant 0\}$ *be a controlled process satisfying* (1.1). *Then this process is optimal if and only if* $\rho^*(Z(t)) = \rho(X(t-)) = \rho(t) \quad \lambda \otimes P$*–a.s., where* $Z(t) = X(t) + Y(t)$, $\rho(t) = \mu(t)/\sigma^2(t)$ *and* ρ^* *is given by* (3.1).

Proof. Let $(X(t), Y(t))$ satisfy (1.1) and the condition $\rho^*(Z(t))=\rho(X(t-))=\rho(t)$, $\lambda\otimes P$–a.s., then following the proof of Proposition 3.3, with $\varepsilon=0$, it follows that $(X(t), Y(t))$ is an optimal process. The proof of the other direction follows from Lemma 4.1. □

The drawback of the above theorem is, that it does not tell us how and when to construct optimal processes with the only apriori knowledge of control sets $\{C(r)\colon 0\leqslant r\leqslant a\}$. But when the function ρ given in (1.5) is monotone increasing, this theorem can be used to show that investing all the wealth in the risky asset is optimal. This strategy was used in Sudderth and Weerasinghe (1991) and was labeled "bold play". With bold play, $Y(t)\equiv 0$ for all $t>0$, $X(0+)=x+y$ and the process will be stopped whenever $X(t)$ becomes 0 or a.

Corollary 4.3. *If ρ is monotone increasing on $[0,a]$ then "bold play" is optimal.*

Proof. Let $(X(t), Y(t))$ be the process corresponding to bold play. Then $(X(0), Y(0))=(x,y)$, $X(0+)=\min\{x+y,a\}$, $\mathrm{d}X(t)=\mu_0(X(t))\,\mathrm{d}t+\sigma_0(X(t))\,\mathrm{d}W(t)$, and $Y(t)\equiv 0$ for all $t>0$, where (μ_0,σ_0) satisfies (1.6).

In this case it is easy to check $Z(t)\equiv X(t)$, $\rho^*(Z(t))=\rho(X(t))$ for all $t>0$, and hence from the previous theorem the result follows. □

Corollary 4.4. *If the collection of control sets $\{C(x)\colon 0\leqslant x\leqslant a\}$ is monotone increasing, then ρ is monotone increasing on $[0,a]$ and consequently "bold play" is optimal.*

The next result gives a sufficient condition in terms of the function ρ for non-existence of an optimal process.

Theorem 4.5. *Let ρ and ρ^* be defined as before. Assume that there exists a point x_0 in $(0,a)$ satisfying the following conditions*:

$$\text{(i)}\quad \rho^*(x_0)>\rho(x_0) \tag{4.3}$$

$$\text{(ii)}\quad \textit{The set } \{y\colon 0\leqslant y<x_0;\ \rho(y)=\rho^*(x_0)\} \textit{ is finite.} \tag{4.4}$$

Then there is no optimal process.

Proof. Let $(X(t), Y(t))$ be an optimal process satisfying Eq. (1.1) and let $Z(t)=X(t)+Y(t)$. Notice that $Z(t)$ is a continuous process governed by

$$\begin{aligned} \mathrm{d}Z(t)&=\mu(t)\,\mathrm{d}t+\sigma(t)\,\mathrm{d}W(t),\\ Z(0)&=x+y. \end{aligned} \tag{4.5}$$

By Lemma 4.1, $\mu(t)/\sigma^2(t)=\rho(t)=\rho(X(t-))=\rho^*(Z(t))$ a.s. $\lambda\otimes P$ where λ is Lebesgue measure on $[0,\infty)$. Therefore using Itô's lemma, for $S^*(Z(t))$ one can easily show that the probability $P[Z(t)$ reaches r before $a]$ for any $0<r<a$ is strictly positive.

By (4.3), there is a $\delta>0$, such that $\rho^*(u)>\rho(u)$ and $\rho^*(u)=\rho^*(x)$ for each u in $[x_0-\delta,x_0]$. The process $Z(t)$ crosses the interval $[x_0-\delta,x_0]$ with positive probability. Now $\rho^*(Z(t))=\rho(X(t-))$ implies that, when $Z(t)$ is in $[x_0-\delta,x_0]$, $X(t)$ takes the values on the finite set $\{y\colon 0\leqslant y<x_0;\ \rho(y)=\rho^*(x_0)\}$. But since $X(t)$ satisfies (1.1),

this implies that $A(t)$ is of unbounded variation on the finite time duration where $Z(t)$ is inside the interval $[x_0-\delta, x_0]$. Hence there is no optimal process. □

The following example does not satisfy the assumptions in Theorem 4.5, but still there is no optimal process.

Example. Assume that ρ is a continuous function on $[0, a]$ satisfying the following conditions. Let $0<\alpha<\beta<a$ and ρ is constant on $[0,\alpha]$, strictly decreasing on $[\alpha,\beta]$, strictly increasing on $[\beta,a]$ and $\rho(a)>\rho(x)$ for all x in $[0,a)$. Then it is easy to check that there is no x_0 in $(0,a)$ satisfying conditions (4.3) and (4.4) in Theorem 4.5.

Suppose that $(X(t), Y(t))$ is an optimal process satisfying (1.1)–(1.3) and let $Z(t)=X(t)+Y(t)$. Then $Z(t)$ is a continuous process satisfying (4.5). Then by Lemma 4.1, $\rho(X(t-))=\rho^*(Z(t))$, almost surely in $\lambda\otimes P$ where λ is the Lebesgue measure on $[0,\infty)$. Let c be the unique point in (β,a) such that $\rho(c)=\rho(x)$ for all x in $[0,\alpha]$. Since $Z(t)$ satisfies (4.5), $Z(\cdot)$ reaches the point c and it crosses the point c infinitely many times with positive probability. But $\rho(X(t-))=\rho^*(Z(t))$ implies that for each upcrossing of Z across the point c, $X(t)$ process jumps from α to c. Therefore $X(t)$ will jump from α to c infinitely many times with positive probability and hence $A(t)$ process will be of unbounded variation. This contradicts assumptions (1.1)–(1.3). Therefore, there is no optimal process.

Our final result gives a sufficient condition in terms of the function ρ for the existence of an optimal process, and the proof is by construction. But we are unable to describe a necessary and sufficient condition in terms of the function ρ for the existence of an optimal process.

Definition 4.6. A function ρ defined on $[0, a]$ is said to have "Property H" if it satisfies
(i) ρ is continuous on $[0, a]$
(ii) The set $\{x\in[0,a]: \rho(x)=\rho^*(x)\}$, where ρ^* is given by (3.1), can be written as a finite union of non-overlapping connected sets $(A_k: k=1,\dots,N)$ satisfying the conditions described below. For convenience we assume that if $1\leqslant i\leqslant j\leqslant N$, then the set A_j lies to the right of A_i.

For each k, $1\leqslant k\leqslant N$, there exist three non-overlapping closed intervals $I_k^{(1)}, I_k^{(2)}$ and $I_k^{(3)}$ satisfying $A_k=I_k^{(1)}\cup I_k^{(2)}\cup I_k^{(3)}$ where if $i<j$, $I_k^{(j)}$ lies to the right of $I_k^{(i)}$, ρ is constant on $I_k^{(1)}$ and $I_k^{(3)}$, and ρ is strictly increasing on $I_k^{(2)}$. We require that $I_k^{(1)}$ and $I_k^{(3)}$ are intervals of positive length, and the left most interval $I_1^{(1)}$ contains zero. But we allow the cases where $I_1^{(1)}=\{0\}$ and $I_N^{(3)}=\{a\}$.

Theorem 4.7. *If the function* ρ *given by* (1.5) *satisfies* (1.6) *and the "property H" given in the above definition, then there is an optimal process.*

Proof. To construct our candidate $(X(t), Y(t))$ for an optimal process, we use the drift and diffusion coefficients $(\mu_0(x), \sigma_0(x))$ given by (1.6). We intend to choose the bounded variation process $A(t)$ so that the process $(X(t), Y(t))$ satisfies the Proposition 3.3 with $\varepsilon=0$, and therefore proving the optimality of the choice of this strategy.

Since the function ρ satisfies the "property H", we can write the set $\{x \in [0,a]: \rho(x) = \rho^*(x)\} = \bigcup_{d=1}^{N} A_j$ where each A_j can be written in the form $A_j = I_j^{(1)} \cup I_j^{(2)} \cup I_j^{(3)}$, as described above. Let $I_j^{(1)} = [a_j, b_j]$, $I_j^{(2)} = [b_j, c_j]$ and $I_j^{(3)} = [c_j, d_j]$. Also let α_j and β_j to be the mid-points of the intervals $I_j^{(1)}$ and $I_j^{(3)}$ respectively. Let $Z(0) = z$ be the initial total wealth and assume that $0 < z < a$. First we consider the case $z \notin \{x \in [0,a]: \rho(x) = \rho^*(x)\}$. Notice that there exist $1 \leqslant k \leqslant N$ such that $d_k < z < a_{k+1}$. (Let $a_{N+1} \equiv a$). Choose $X(0+) = \beta_k$ and $Y(0+) = z - \beta_k$ and let $X(t)$ to be the diffusion with drift and diffusion coefficients $\mu_0(\cdot)$ and $\sigma_0(\cdot)$ respectively, with reflecting barriers at the end points c_k and d_k of $I_k^{(3)}$. Then $Y(t) = Y(0+) + L^X(d_k) - L^X(c_k)$ where $L^X(c_k)$ and $L^X(d_k)$ are the local times of $X(t)$ process at c_k and d_k, respectively. Eventually, the joint process $(X(t), Y(t))$ reaches $(c_k, 0)$ or $(d_k, d_{k+1} - d_k)$. If it reaches $(c_k, 0)$ then it cannot reflect at c_k and $X(t)$ process continues as a diffusion with one reflecting barrier at d_k, and it continues until $(X(t), Y(t))$ reaches $(a_k, 0)$ or $(d_k, \alpha_{k+1} - d_k)$.

Notice that if $k = 1$, $(a_1, 0) \equiv (0,0)$ in which case, if $(X(t), Y(t)) = (0,0)$ then the process should be stopped. Similarly, if $k = N$, we run the process $(X(t), Y(t))$ until it reaches $(a_k, 0)$ or $(d_k, a - d_k)$, respectively, and if $(X(t), Y(t)) = (d_k, a - d_k)$ then the total wealth $X(t) + Y(t) = a$ and the process will be stopped.

In the case if $z \in \{x \in [0,a]: \rho(x) = \rho^*(x)\}$ then $z \in A_k$ for some k and hence $a_k \leqslant z \leqslant d_k$. In this case we make $X(0+) = z$, $Y(0+) = 0$ and let $X(t)$ to be the diffusion process with coefficients $\mu_0(\cdot)$ and $\sigma_0(\cdot)$ and reflecting barriers at a_k and d_k and the strategy is same as described above. A straightforward calculation shows that $\rho(X(t)) = \rho^*(Z(t))$ where $Z(t) = X(t) + Y(t)$ and ρ^* is given by (3.1). Hence it satisfies Eqs. (3.3) and (3.4) with $\varepsilon = 0$, and Proposition 3.3 implies that $(X(t), Y(t))$ is an optimal process. $\square$

5. Examples

The following two examples illustrate somewhat intuitive ε-optimal policies one can construct by using reflecting diffusions and their local times. First, we define the scale function $S(\cdot)$ by

$$S(x) = \int_0^x \mathrm{e}^{-2\int_0^r \rho(u)\,\mathrm{d}u}\,\mathrm{d}r \tag{5.1}$$

where $\rho(\cdot)$ is given by (1.5).

Example 1′. Let us assume that

$$\begin{aligned} &\text{(i)} \quad \rho \text{ is continuous and satisfies (1.6), and} \\ &\text{(ii)} \quad \rho(0) = \max_{[0,a]} \rho(x). \end{aligned} \tag{5.2}$$

We can apply Theorem 4.2 to show that there are no optimal strategies. If $(X,Y) \in \Sigma(x,y)$ is an optimal process, then by Theorem 4.2, $\rho((X(t-)) = \rho^*(Z(t))$ for all t, but $\rho^*(Z) = \rho(0)$ for all z and hence $X(t) \equiv 0$ for all $t > 0$. Obviously this is a contradiction and hence there are no optimal processes.

To describe ε-optimal processes for this example, take any $(x,y)\in E$ with $x+y<a$. Pick $\varepsilon>0$ so that $0<\varepsilon<a-y$. If $0\leqslant x\leqslant\varepsilon$, define the X_ε process to be the reflecting diffusion with drift coefficient $\mu_0(\cdot)$ and diffusion coefficient $\sigma_0(\cdot)$ with instantaneous reflection at 0 and ε. The Y_ε process decreases as local time at 0 whenever X_ε is at 0, while Y_ε increases as local time at ε whenever X_ε is at ε. If $x>\varepsilon$, initially the X_ε process jumps to ε and then follow the same procedure as above. First, let $0\leqslant x\leqslant\varepsilon$.

$$\mathrm{d}X_\varepsilon(t)=\mu_0(X_\varepsilon(t))\,\mathrm{d}t+\sigma_0(X_\varepsilon(t))\,\mathrm{d}W(t)+\mathrm{d}L^0(t)-\mathrm{d}L^\varepsilon(t)$$

and

$$\mathrm{d}Y_\varepsilon(t)=\mathrm{d}L^\varepsilon(t)-\mathrm{d}L^0(t) \tag{5.3}$$

with

$$(X_\varepsilon(0),Y_\varepsilon(0))=(x,y).$$

In this strategy introduce the quitting time τ by $\tau=\inf\{t\geqslant 0\colon X_\varepsilon(t)+Y_\varepsilon(t)=a \text{ or } 0\}$. Notice that $X_\varepsilon(t)$ and $Y_\varepsilon(t)$ are non-negative if $0\leqslant t\leqslant\tau$. Furthermore, the player spends $Y_\varepsilon(t)$ as local time of X_ε at 0 to keep X_ε non-negative and the player increases $Y_\varepsilon(t)$ as the local time of X_ε at ε.

The following proposition proves that the corresponding $(X_\varepsilon,Y_\varepsilon)$ process is an ε-optimal process.

Proposition 5.1. (i) *Let* $y+\varepsilon<a$. *Define* $(X_\varepsilon,Y_\varepsilon)\in\Sigma(x,y)$ *as described in* (5.3). *Introduce the stopping time* τ *by*

$$\tau=\inf\{t\geqslant 0\colon X_\varepsilon(t)+Y_\varepsilon(t)=a \text{ or } 0\}.$$

Then

$$\text{(i)}\quad P_{(x,y)}[X_\varepsilon(\tau)+Y_\varepsilon(\tau)=a] =\begin{cases} U_\varepsilon(x,y)/U_\varepsilon(\varepsilon,a-\varepsilon) & \text{if } 0\leqslant x\leqslant\varepsilon,\\ U_\varepsilon(\varepsilon,x+y-\varepsilon)/U_\varepsilon(\varepsilon,a-\varepsilon) & \text{if } \varepsilon<x<a-y,\end{cases} \tag{5.4}$$

where

$$U_\varepsilon(x,y)=S(x)\mathrm{e}^{-\left(\frac{1-S'(\varepsilon)}{S(\varepsilon)}\right)y}+\left(\frac{S(\varepsilon)}{1-S'(\varepsilon)}\right)\left(1-\mathrm{e}^{-\left(\frac{1-S'(\varepsilon)}{S(\varepsilon)}\right)y}\right). \tag{5.5}$$

(ii) *For each* $(x,y)\in E$, *the above described* $(X_\varepsilon,Y_\varepsilon)$ *process is an* ε*-optimal strategy.*

Proof. Let $0\leqslant x\leqslant\varepsilon$. By direct computation one can check that U_ε solves

$$\frac{1}{2}\,\sigma_0^2(x)\,\frac{\partial^2U_\varepsilon}{\partial x^2}(x,y)+\mu_0(x)\frac{\partial U_\varepsilon}{\partial x}(x,y)=0 \quad \text{for } (x,y)\in E,$$

$$\frac{\partial U_\varepsilon}{\partial x}(0,y)-\frac{\partial U_\varepsilon}{\partial y}(0,y)=0 \quad 0\leqslant y\leqslant a,$$

$$\frac{\partial U_\varepsilon}{\partial x}(\varepsilon,y)-\frac{\partial U_\varepsilon}{\partial y}(\varepsilon,y)=0 \quad 0\leqslant y\leqslant a,$$

and $U_\varepsilon(0,0)=0$.

Next, consider the $(X_\varepsilon, Y_\varepsilon)$ process and introduce the stopping time $\tau = \inf\{t > 0: X_\varepsilon(t) + Y_\varepsilon(t) = a \text{ or } 0\}$. Notice that if $X_\varepsilon(\tau) + Y_\varepsilon(\tau) = 0$ then $(X_\varepsilon(\tau), Y_\varepsilon(\tau)) = (0, 0)$ and if $X_\varepsilon(\tau) + Y_\varepsilon(\tau) = a$ then $(X_\varepsilon(\tau), Y_\varepsilon(\tau)) = (\varepsilon, a - \varepsilon)$ since $y + \varepsilon < a$.

We apply Itô's formula to $U_\varepsilon(X_\varepsilon(t \wedge \tau), Y_\varepsilon(t \wedge \tau))$ and taking the expectation we get

$$E_{(x,y)}[U_\varepsilon(X_\varepsilon(t \wedge \tau), Y_\varepsilon(t \wedge \tau))] = U_\varepsilon(x, y).$$

Since U_ε is bounded, we let $t \to \infty$ to derive

$$E_{(x,y)}[U_\varepsilon(X_\varepsilon(\tau), Y_\varepsilon(\tau))] = U_\varepsilon(x, y).$$

and

$$E_{(x,y)}[U_\varepsilon(X_\varepsilon(\tau), Y_\varepsilon(\tau))] = U_\varepsilon(\varepsilon, a - \varepsilon) P_{(x,y)}[X_\varepsilon(\tau) + Y_\varepsilon(\tau) = a].$$

Hence $P_{(x,y)}[X_\varepsilon(\tau) + Y_\varepsilon(\tau) = a] = U_\varepsilon(x, y)/U_\varepsilon(\varepsilon, a - \varepsilon)$.

The formula for the case $\varepsilon < x < a - y$ follows from the definition of the process, since $(X_\varepsilon(0+), Y_\varepsilon(0+))$ becomes $(\varepsilon, x + y - \varepsilon)$.

To prove part (ii), one can employ the fact $\lim_{\varepsilon \to 0} (1 - S'(\varepsilon))/S(\varepsilon) = 2\rho(0)$ where S is given by (5.1) to show $\lim_{\varepsilon \to 0} P_{(x,y)}[X_\varepsilon(\tau) + Y_\varepsilon(\tau) = a] = (S^*(x + y))/S^*(a)$, where S^* is given by (3.2). Hence $(X_\varepsilon, Y_\varepsilon)$ is an ε-optimal process. □

Example 2. Consider the case where for some c in $[0, a]$, $\rho(0) = \max_{[0,c]} \rho(u)$, ρ is increasing on $[c, a]$ and $\rho(c) = \rho(0)$. Furthermore, we assume ρ is continuous and satisfies (1.6). Notice $\rho(a) > \rho(c) = \rho(0)$, and hence this case is not covered before. The situation where ρ is decreasing on $[0, b]$ and is increasing on $[b, a]$ for some b, with $\rho(a) > \rho(0)$ is also included in this case.

Take $\varepsilon > 0$ small enough. We now give a candidate for an ε-optimal process. Let $(x, y) \in E$ with $x + y + \varepsilon \leqslant c$. First let the X-process jump to zero and increase the Y-process to $x + y$. Then follow the strategy given in Example 1 to reach c. Once c is reached, the value of the Y process will be zero, and then let the X process run according to the diffusion with coefficients μ_0 and σ_0 until it reaches $c - \varepsilon$ or the goal a whichever happens first. In the case that it reaches $c - \varepsilon$, then the player should jump back to zero so that the X process will take value zero and the Y process will be increased to $c - \varepsilon$. Now again the player is at $(0, c - \varepsilon)$ and so he can continue as initially. If $x + y > c - \varepsilon$, then the player jumps to $x + y$ by making the Y process 0 and letting the X process run according to the diffusion with coefficients μ_0 and σ_0 until it reaches $c - \varepsilon$ or the goal a whichever happens first. Then continue as described above.

To compute the value function, we introduce a scale function $S_0(x)$ given by

$$\begin{aligned} S_0(x) &= \int_0^x e^{-2\rho(0)r}\, dr = \frac{1}{2\rho(0)}(1 - e^{2\rho(0)x}) \qquad \text{if } \rho(0) \neq 0, \\ &= x \qquad\qquad\qquad\qquad\qquad\qquad\qquad\quad \text{if } \rho(0) = 0. \end{aligned} \tag{5.6}$$

The following proposition can be proved similar to Proposition 5.1, by using the Markov property of the above mentioned strategy and Itô's formula. Therefore, we state the proposition without a proof. Similarly, one can construct ε-optimal processes for many other classes of functions ρ including the case where ρ is increasing on $[0, c]$ and $\rho(c) = \sup_{[0,a]} \rho(x)$ for some c in $(0, a)$.

Proposition 5.2. *Let $(x,y)\in E$, $x+y<a$ and $\varepsilon>0$ with $0<c-\varepsilon<c+\varepsilon<a$. Let $(X_\varepsilon, Y_\varepsilon)\in\Sigma(x,y)$ be the process which represents the above mentioned strategy. Introduce $Q_\varepsilon(x,y)$ by*

$$Q_\varepsilon(x,y)=P[X_\varepsilon(t)+Y_\varepsilon(t)=a \quad \textit{before } 0 \textit{ for some } t>0|(X_\varepsilon(0),Y_\varepsilon(0))=(x,y)]. \tag{5.7}$$

Then

$$Q_\varepsilon(x,y)=\frac{U_\varepsilon(0,x+y)}{U_\varepsilon(\varepsilon,c+\varepsilon)}\cdot\frac{(S(c)-S(c-\varepsilon))}{(S(a)-S(c-\varepsilon))} \times\left(1-\frac{(S(a)-S(c))U_\varepsilon(0,c-\varepsilon)}{(S(a)-S(c-\varepsilon))U_\varepsilon(\varepsilon,c-\varepsilon)}\right)^{-1} \tag{5.8}$$

for $x+y\leqslant c-\varepsilon$ and

$$Q_\varepsilon(x,y)=\frac{S(x+y)-S(c-\varepsilon)}{S(a)-S(c-\varepsilon)}+\frac{S(a)-S(x+y)}{S(a)-S(c-\varepsilon)}Q_\varepsilon(0,c-\varepsilon) \quad \textit{if } x+y>c-\varepsilon, \tag{5.9}$$

where $Q_\varepsilon(0,c-\varepsilon)$ is given by (5.8), *S and U_ε are given by* (5.1) *and* (5.5), *respectively.*

Moreover,

$$\lim_{\varepsilon\to 0} Q_\varepsilon(x,y)=\frac{S^*(x+y)}{S^*(a)}, \tag{5.10}$$

where S^* is given by (3.2) and $(X_\varepsilon, Y_\varepsilon)$ is an ε-optimal process.

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